Type of mapping —

1. Translation
2. Rotation
3. Magnification

Inversion $= \frac{1}{z} = S(z)$

1. <u>Translation</u> — $T(z) = z + b$, $b \neq 0$

2. <u>Magnification</u> (Dilation) — $M(z) = az$, a is real, $a > 0$

3. <u>Rotation</u> — $R(z) = e^{i\theta} z$ where $|a| = 1$

Effect of Translation, Rotation & Magnification on a geometric figure —

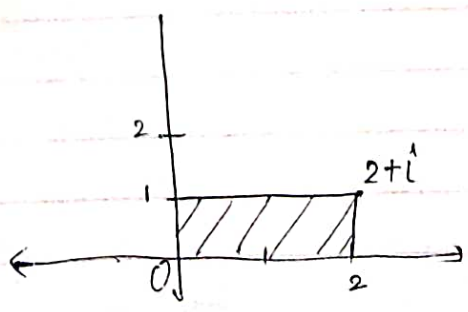
1. Translation doesn't change the shape & size of a geometrical figure.
2. Rotation doesn't change the shape and size of a geometric figure.

* 3. Magnification doesn't change the shape But it can change the size.

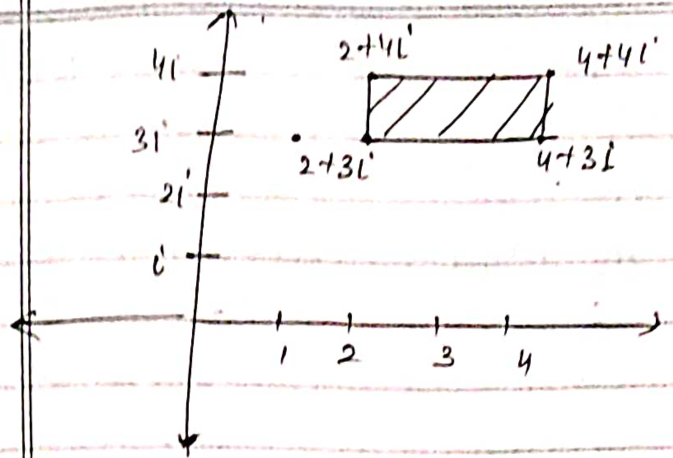
Quest^n Find Image of a Retangle with vertices $(0, 2, 2+i, i)$ under the mappings

1. $T(z) = z + 2 + 3i$
2. $M(z) = 2z$
3. $R(z) = e^{i\pi/2} z$

$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\pi/2$
$= 0 + i = i$

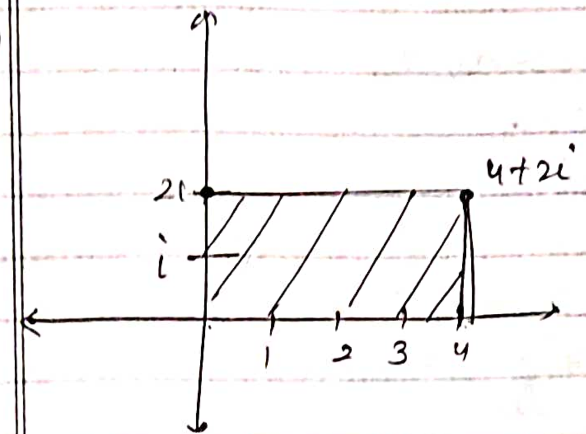


(1)



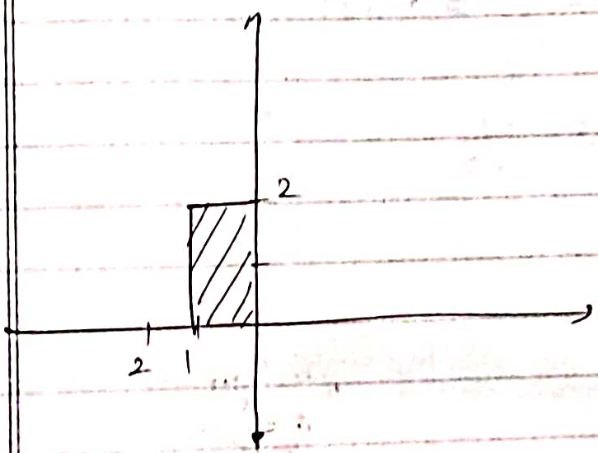
$T(z) = z + 2 + 3i$
$T(0) = 2 + 3i = (2,3)$
$T(2) = 4 + 3i = (4,3)$
$T(i) = 2 + 4i = (2,4)$
$T(2+i) = 4 + 4i$
$= (4,4)$

(2)



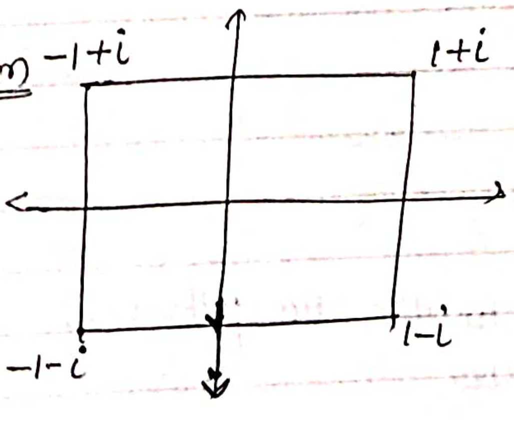
$M(z) = 2z$
$M(0) = 0$
$M(2) = 4$
$M(i) = 2i$
$M(2+i) = 4+2i$

(3)



$R(z) = e^{i\pi/2} z = (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) z$
iz
$\Rightarrow R(z) = i \cdot z$
$R(0) = 0.$
$R(2) = 2i$
$R(i) = -1.$
$R(2+i) = -1+2i$

Question



$R(z) = e^{i\pi/2} z$

$R(z) = iz$

$R(1+i) = -1+i$

$R(1-i) = i+1$ — fig remains same

$R(-1-i) = -i+1$

$R(-1+i) = -i-1$

Now Take $R(z) = e^{i\pi/4} z.$

$= (\cos \pi/4 + i \sin \pi/4) z$

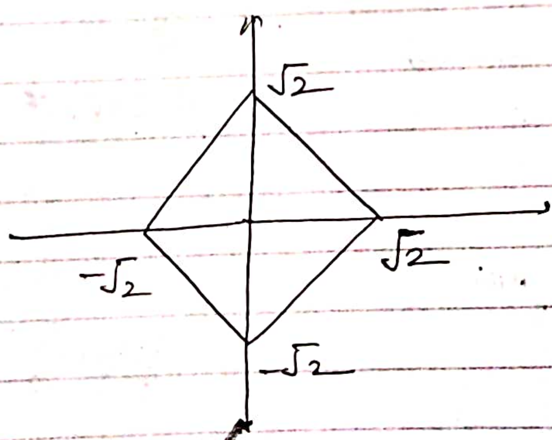
$= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) z.$

$R(1+i) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{2i}{\sqrt{2}} = \sqrt{2}\, i$

$R(1-i) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$R(-1-i) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\sqrt{2}\, i$

$R(-1+i) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$



for $\theta = \pi/4$

✓ In Rotation – we have to rotate the plane. not the figure.

Now $w = z+b$
$w = az$, $a>0$ → Magnification
$w = az$, $|a|=1$ → Rotation

$w = az$, a any complex no.

$= re^{i\theta} . z$

Magnif^n → Rotation

So Here $w = az$. a arbitrary complex no.
is called Magnification & Rotation.

→ In rotation mapping e^{θ} — angle tell us by how much angle the plane will rotate.

Linear mapping →
$f(z) = az + b$ → Translation
magnif^n or Rotation

ie Linear map^g is a combination of magnification, rotation & Translation.

Commutative Property of T, M & R →

1. $ToM \neq MoT$
2. $ToR \neq RoT$
3. $MoR = RoM$.

T $T(z) = z+b$
M $T(z) = az$ $a \neq 0$
R $T(z) = ze^{i\theta} = az$

Pf Let $T(z) = z+b$
$M(z) = az$
$R(z) = e^{i\theta} z$

$ToM(z) = T[M(z)]$
$= T(az)$
$= az+b$

$MoT = MoT(z)$
$= M(z+b) = a(z+b) = az+ab$

(i) $T \circ M(z) = T(M(z)) = T(az)$
$= az + b.$

$M \circ T(z) = M(T(z)) = M(z+b) = az + ab$

$\therefore T \circ M \neq M \circ T.$

(ii) $T \circ R(z) = T(R(z)) = T(e^{i\theta} z) = e^{i\theta} z + b$

$R \circ T(z) = R\,T(z) = R(z+b) = e^{i\theta}(z+b)$

$\therefore T \circ R \neq R \circ T.$

(iii) $M \circ R(z) = M(R(z)) = M(e^{i\theta} z) = a e^{i\theta} z$

$R \circ M(z) = R(M(z)) = R(az) = a e^{i\theta} z$

$\therefore M \circ R = R \circ M$

<u>NOTE</u> – ① A linear mapping doesnot change the shape of a geometrical figure but can change the size of the geometric figure.

② To find image of circle. find the centre and any point from the boundary.

Pg 360

Bilinear transformation / Möbius transformation / Linear fractional Transformation

$$w = f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

If $ad-bc=0$ Then $ad=bc$

$$\frac{a}{c} = \frac{b}{d} = K \text{ say}$$

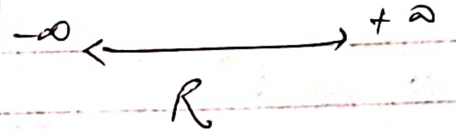
$$\Rightarrow a = cK, \quad b = dK$$

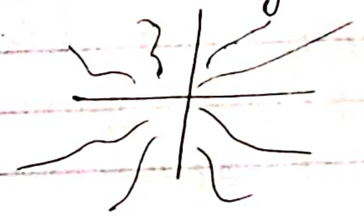
$$\therefore w = f(z) = \frac{cKz+dK}{cz+d} = \frac{K(cz+d)}{cz+d} = K.$$

$= \text{constant}$

Results -

1. There are infinite dir^{n}s in complex plane
In real line we have only 2 dir^n.

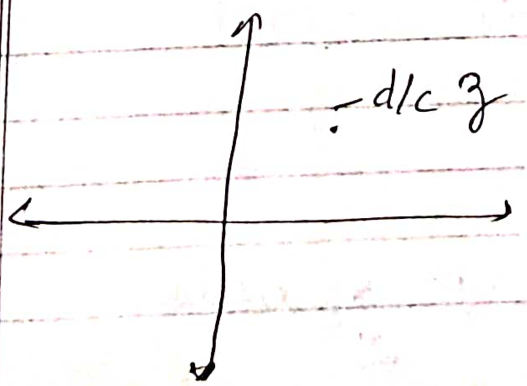


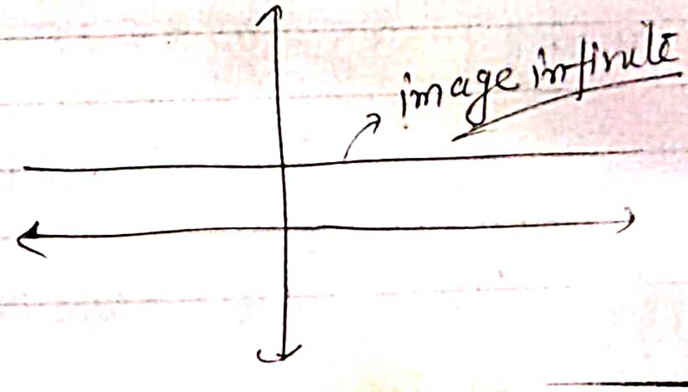


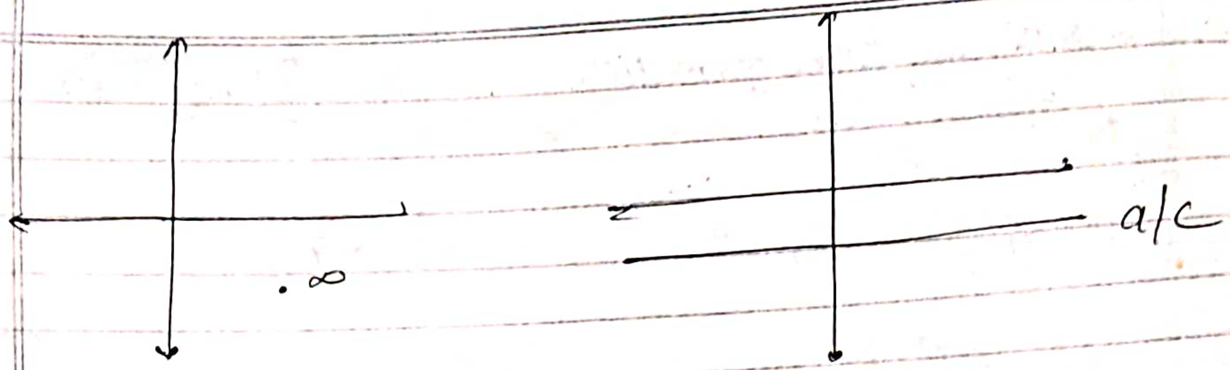
2. Riemann - There Steriography Projection - There is only one infinity in complex plane.
i.e $+\infty$

(or) All infinity corresponds to one point.

at $z = -d/c$ $\quad f(z) = \infty$







** Most Important Result –

A Bilinear transformation transforms circles or straight lines into circles or straight lines. According to the following rules –

(i) If pole $z = -d/c$ lie on given line or circle
Then image is a line.

(ii) If pole $z = -d/c$ doesn't lie on the given line or circle
Then image is a circle.

or $L \longrightarrow L$ or $L \longrightarrow C$
$C \longrightarrow L$ $C \longrightarrow C$

The Three forms of equations of circle –

1. $|z - z_0| = r$, centre z_0, radius $= r$

2. $z\bar{z} + \bar{a}z + a\bar{z} + c = 0$, c is real const.

3. $\left|\dfrac{z-p}{z-q}\right| = k$, $k \neq 1$

Here p & q are inverse pt of circle.

Remark – (1) Two sides of the line ie interior or exterior are transformed onto the 2 sides ie interior ie exterior or

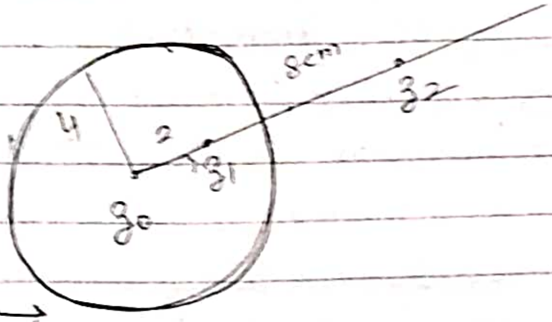
(2) To check the situation we usually take help of a Test point.

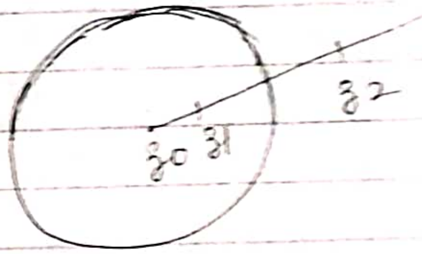
(5) is a special form for solving in Bilinear Eq^n.

Def^n – Inverse point w.r. to a circle →

Two points z_1 & z_2 are said to be inverse points w.r.t a circle $|z - z_0| = r$ if

1. z_1 & z_2 lie on the same side of centre z_0
2. $|z_1 - z_0|\,|z_2 - z_0| = r^2$





Result →
Inverse point of the center of the circle is infinity.

Result →
point on the Boundary is inverse point of itself.

Formula for finding Inverse point –

Inverse point of a point 'a' w.r.t the circle $|z - z_0| = r$ is
$$\boxed{z_0 + \frac{r^2}{\overline{a} - \overline{z_0}}}$$

ie Centre + $\dfrac{(\text{radius})^2}{\overline{(\text{given pt})} - \overline{(\text{centre})}}$

Qu^n find inverse pt of $z=2$ & $z=1+i$ w.r.t circle $|z|=1$

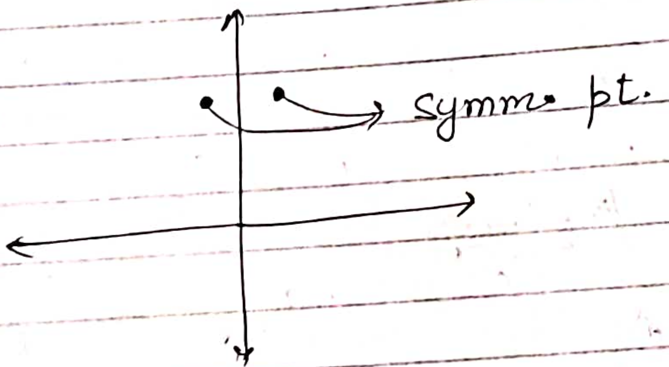
centre $=0$, $r=1$

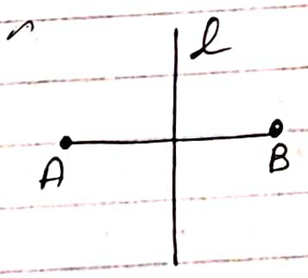
w.r.t 2 $= 0 + \frac{1}{2-0} = \frac{1}{2}$

w.r.t $1+i$ $= 0 + \frac{1}{\overline{1+i}-0} = \frac{1}{1-i}$

Symmetrical points w.r.t a line →

Two points are said to be symmetrical point w.r.t a line if they are mirror images of each other when the mirror placed at the line.





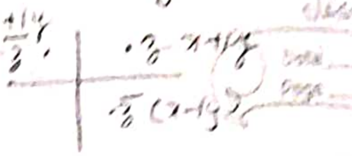
line l is $\perp$ bisector of the segment A & B

Note
Equation of a straight line having p & q as its symmetrical points is given by

$$\left|\frac{z-p}{z-q}\right| = 1$$

Result — If 2 pts are symm. w.r.t x axis Then they are conjugate of each other

→ If 2 pt are symm abt. y-axis Then They are -ve conjugate of each other



Most Important Result continues....

→ Transform

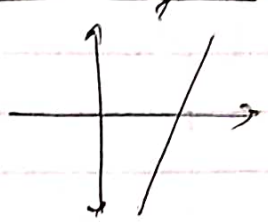
Last four cases are –

(1) if Line → line
Then symm pt → symm pt

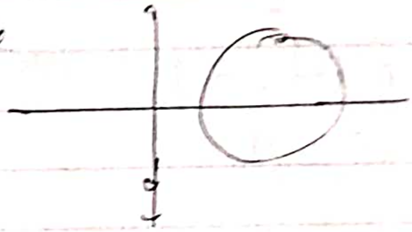
(2) if Line → circle
Then symm pt → Inverse pt

(3) If circle → line
then Inverse pt → Symm pt

(4) If circle → circle
Then Inverse pt → Inverse pt.



line → circle



ie When line transforms into circle it means pt on line will lie on bdry of circle & pt. lie on one side of line lie inside the circle ie in Interior & pts on other side lie exterior of the circle.

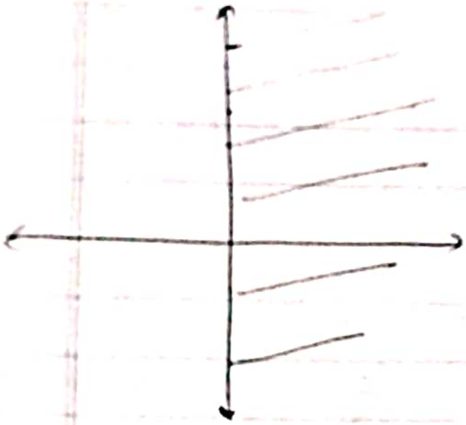
→ we can't Draw. But can imagine
(or) By using concept of countability

To check which part lie inside / outside the circle. Take a pt from that part if that pt lie inside the circle then that portion lie inside If not then that portion become exterior part of circle.

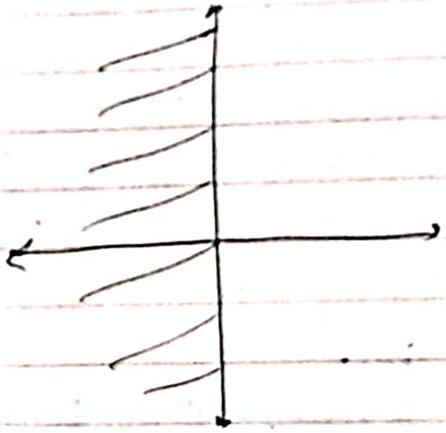
Note Two sides of the line (or) interior or exterior are transformed onto the two sides or interior or exterior.

Remark - To clear the situation we usually take help of a test-point.

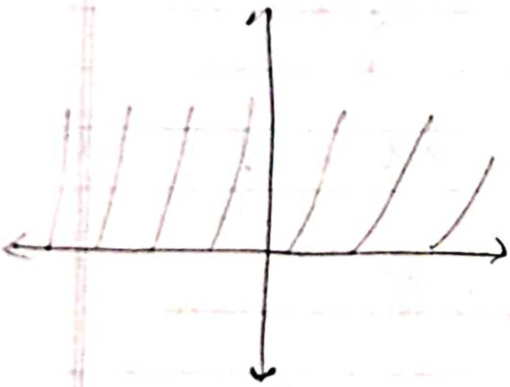
Half-plane



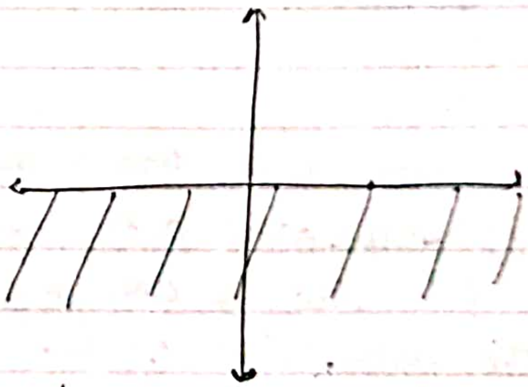
Right half plane



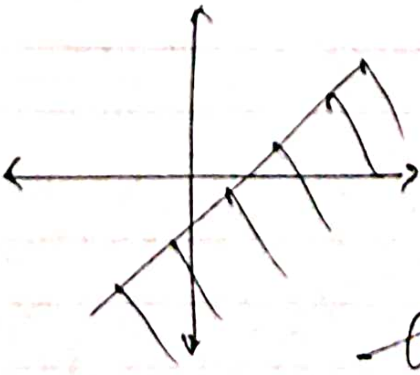
left half plane



upper half plane



lower half plane



-half plane

$$\frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)} \qquad \frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)}$$

classmate Date Page

$$\frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)}$$

Cross Ratio-

Cross ratio of four complex numbers z_1, z_2, z_3, z_4 defined as

$$\frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)} \qquad \frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)}$$

$$\frac{z_2-z_3}{z_1-z_2} \quad \frac{z_4-z_1}{z_3-z_4} \qquad \frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)}$$

Result- Cross ratio is invariant under a bilinear transformation. (08)

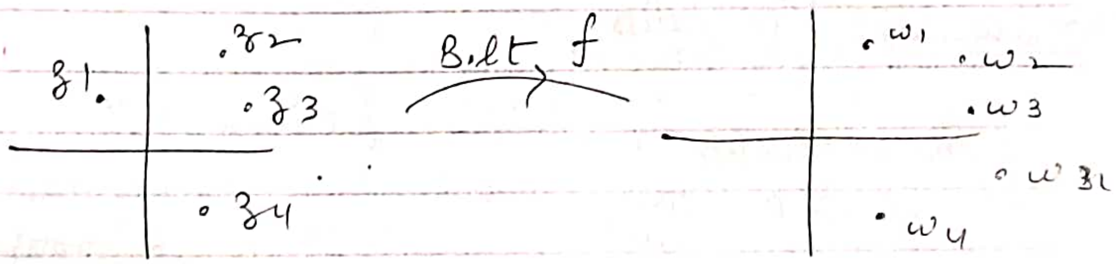
Let $f(z) = \dfrac{az+b}{cz+d}$ is a bilinear transformation

& z_1, z_2, z_3, z_4 be any four points.

Let w_1, w_2, w_3, w_4 be their images under f.

Then

$$\frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)} = \frac{(w_2-w_3)(w_4-w_1)}{(w_1-w_2)(w_3-w_4)}$$



Result fixed point -

If $f(z)=z$ then z is said to be fixed pt. of f.

$\boxed{\text{Cases for fixed point}} \rightarrow$

(1) L.T-

L.T - $az+b$

To find fixed point

$$f(z)=z$$
$$az+b=z$$
$$z=\frac{b}{1-a}, \quad a\neq 1$$

$\Rightarrow$ Linear mapping has only 1 fixed point

$$\boxed{z = \frac{b}{1-a}}$$

If $a=1$ Then $z + b = z$

$b = 0$

Then map^n become Rotational.

Constant mapping —

$$f(z) = k$$

constant mapping has exactly one fixed point.

Identity mapping —

$$f(z) = z$$

Every point is a fixed point

Non-Identity Translation —

$$T(z) = z + b$$

(non-Identity translation $\Rightarrow b \neq 0$.
$\because$ If $b=0$ Then $T(z) = z$ is Identity mapping)

$$T(z) = z$$
$$\Rightarrow z + b = z$$
$$\Rightarrow b = 0$$ Contradiction.

$\therefore$ No fixed point.

Non identity Rotation — , Refer

$$T(z) = az \quad , |a| = 1$$

No identity means $a \neq 0$

when $a=1$, it will become identity map.

& every pt will be fixed pt.

zero is the only fixed point.

$$T(z) = z$$
$$az = z$$
$$az - z = 0$$
$$z(a-1) = 0$$
$$\Rightarrow z = 0$$

Magnification –

$$T(z) = az, \quad a > 0$$
$$a \neq 1$$

$z = 0$ is only fixed point

Bilinear mapping –

$$f(z) = \frac{az+b}{cz+d}$$

$$\frac{az+b}{cz+d} = z$$

$$\Rightarrow cz^2 + dz - az - b = 0$$
$$\Rightarrow cz^2 - (a-d)z - b = 0.$$

roots $= \dfrac{a-d \pm \sqrt{(a-d)^2 + 4bc}}{2c}$

$\Rightarrow$ $f(z)$ has one or two fixed point.
according as
$(a-d)^2 + 4bc$ is zero or non zero.
If zero $\Rightarrow$ 1 fixed point
If non zero $\Rightarrow$ 2 fixed point.

Note – A linear mapping has exactly one fixed point provided it is not a translation.

Identity map is a special kind of Bilinear map
$a = d = 1, \quad b = c = 0$

Table

S. No	Name of mapping	Expression	No. of fixed point
1	Constant map	$f(z)=k$	One, namely k.
2	Identity map	$f(z)=z$	Every pt is a fixed pt.
3	Translation (non-identity)	$T(z)=z+b$, $b\neq 0$	zero. No point is fixed pt
4	Rotation (non-identity)	$T(z)=az$, $\lvert a\rvert=1$, $a\neq 1$	One namely 0.
5	Magnification	$T(z)=az$, $a>0$	One namely 0.
6	Linear	$f(z)=az+b$, $a\neq 1$	One i.e. $z=\frac{b}{1-a}$, $a\neq 1$
7	Bilinear	$f(z)=\frac{az+b}{cz+d}$, $ad-bc\neq 0$	$\boxed{\text{One}}$ if $(a-d)^2+4bc=0$ namely $\frac{a-d}{2c}$; $\boxed{\text{Two}}$ if $(a-d)^2+4bc\neq 0$ namely $\frac{a-d\pm\sqrt{(a-d)^2+4bc}}{2c}$